Chapter 1

RELATIONS & FUNCTIONS

Be Happy. It's Maths time now!

RECAPITULATION & RELATIONS

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. TYPES OF INTERVALS

a) **Open interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying strictly between a and b is called an *open interval*. It is denoted by $]a,b[$ or (a,b) i.e., $\{x \in \mathrm{R} : a < x < b\}$.

b) **Closed interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes both a and b as well is known as a *closed interval*. It is denoted by $[a,b]$ i.e., $\{x \in \mathrm{R} : a \le x \le b\}$.

c) **Open Closed interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it excludes a and includes only b is known as an *open closed interval*. It is denoted by $]a,b]$ or $(a,b]$ i.e., $\{x \in \mathrm{R} : a < x \le b\}$.

d) **Closed Open interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes only a and excludes b is known as a *closed open interval*. It is denoted by $[a,b[$ or $[a,b)$ i.e., $\{x \in \mathrm{R} : a \le x < b\}$.

🕮 RELATIONS

02. Defining the Relation: A relation R, from a non-empty set A to another non-empty set B is mathematically defined as an arbitrary subset of $\mathrm{A} \times \mathrm{B}$. Equivalently, any subset of $\mathrm{A} \times \mathrm{B}$ is a relation from A to B.

Thus, R is a relation from A to B $\Leftrightarrow \mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$

$$\Leftrightarrow \mathrm{R} \subseteq \{(a,b) : a \in \mathrm{A},\ b \in \mathrm{B}\}.$$

Illustrations:

a) Let $\mathrm{A} = \{1,2,4\}, \mathrm{B} = \{4,6\}$. Let $\mathrm{R} = \{(1,4),(1,6),(2,4),(2,6),(4,6)\}$. Here $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$ and therefore R is a relation from A to B.

b) Let $\mathrm{A} = \{1,2,3\}, \mathrm{B} = \{2,3,5,7\}$. Let $\mathrm{R} = \{(2,3),(3,5),(5,7)\}$. Here $\mathrm{R} \not\subset \mathrm{A} \times \mathrm{B}$ and therefore R is not a relation from A to B. Since $(5,7) \in \mathrm{R}$ but $(5,7) \notin \mathrm{A} \times \mathrm{B}$.

c) Let $\mathrm{A} = \{-1,1,2\}, \mathrm{B} = \{1,4,9,10\}$. Let $a \mathrm{R} b$ means $a^2 = b$ then, $\mathrm{R} = \{(-1,1),(1,1),(2,4)\}$.

☞ **Note the followings :**

- *A relation from A to B is also called a relation from A into B.*
- $(a,b) \in R$ *is also written as* aRb *(read as **a is R related to b**).*
- *Let A and B be two non-empty finite sets having p and q elements respectively. Then* $n(A \times B) = n(A).n(B) = pq$. *Then total number of subsets of* $A \times B = 2^{pq}$. *Since each subset of* $A \times B$ *is a relation from A to B, therefore **total number of relations from A to B is given as*** 2^{pq}.

03. DOMAIN & RANGE OF A RELATION

a) **Domain of a relation:** Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in \mathrm{A}$ such that $(a,b) \in \mathrm{R}$ for some $b \in \mathrm{B}$. Domain of R is precisely written as $Dom.(R)$ symbolically.

Thus, $\text{Dom.(R)} = \{a \in \mathrm{A} : (a,b) \in \mathrm{R} \text{ for some } b \in \mathrm{B}\}$.

That is, the domain of R is **the set of first component of all the ordered pairs which belong to R.**

b) **Range of a relation:** Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in \mathrm{B}$ such that $(a,b) \in \mathrm{R}$ for some $a \in \mathrm{A}$.

Thus, Range of $\mathrm{R} = \{b \in \mathrm{B} : (a,b) \in \mathrm{R} \text{ for some } a \in \mathrm{A}\}$.

That is, the range of R is the set of second components of all the ordered pairs which belong to R.

c) **Codomain of a relation:** Let R be a relation from A to B. Then B is called the codomain of the relation R. So we can observe that codomain of a relation R from A into B is the set B as a whole.

Illustrations:

a) *Let* $A = \{1,2,3,7\}, B = \{3,6\}$. *Let aRb means* $a < b$.

Then we have $R = \{(1,3),(1,6),(2,3),(2,6),(3,6)\}$.

Here $Dom.(R) = \{1,2,3\}$, *Range of* $R = \{3,6\}$, *Codomain of* $R = B = \{3,6\}$.

b) *Let* $A = \{1,2,3\}$, $B = \{2,4,6,8\}$. *Let* $R_1 = \{(1,2),(2,4),(3,6)\}$, $R_2 = \{(2,4),(2,6),(3,8),(1,6)\}$. *Then both* R_1 *and* R_2 *are relations from A to B because* $R_1 \subseteq A \times B$, $R_2 \subseteq A \times B$. *Here* $Dom.(R_1) = \{1,2,3\}$, *Range of* $R_1 = \{2,4,6\}$; $Dom.(R_2) = \{2,3,1\}$, *Range of* $R_2 = \{4,6,8\}$.

04. TYPES OF RELATIONS FROM ONE SET TO ANOTHER SET

a) **Empty relation:** A relation R from A to B is called an empty relation or a void relation from A to B if $\mathrm{R} = \phi$.

For example, *let* $A = \{2,4,6\}, B = \{7,11\}$.

Let $R = \{(a,b) : a \in A, b \in B \text{ and } a - b \text{ is even}\}$. *Here R is an empty relation.*

b) **Universal relation:** A relation R from A to B is said to be the universal relation if $\mathrm{R} = \mathrm{A} \times \mathrm{B}$.

For example, *let* $A = \{1,2\}, B = \{1,3\}$. *Let* $R = \{(1,1),(1,3),(2,1),(2,3)\}$. *Here* $R = A \times B$, *so relation R is a universal relation.*

05. RELATION ON A SET & ITS VARIOUS TYPES

A relation R from a non-empty set A into itself is called a relation on A. In other words if A is a non-empty set, then a subset of $\mathrm{A} \times \mathrm{A} = \mathrm{A}^2$ is called a relation on A.

Illustrations:

Let $A = \{1,2,3\}$ *and* $R = \{(3,1),(3,2),(2,1)\}$. *Here R is relation on set A.*

NOTE If A be a finite set having *n* elements then, number of relations on set A is $2^{n \times n}$ *i.e.,* 2^{n^2}.

a) **Empty relation:** A relation R on a set A is said to be empty relation or a void relation if $\mathrm{R} = \phi$. In other words, a relation R in a set A is empty relation, if no element of A is related to any element of A, i.e., $\mathrm{R} = \phi \subset \mathrm{A} \times \mathrm{A}$.

For example, *let* $A = \{1,3\}$, $R = \{(a,b) : a \in A, b \in A \text{ and } a + b \text{ is odd}\}$. *Here R contains no element, therefore it is an empty relation on set A.*

b) Universal relation: A relation R on a set A is said to be the universal relation on A if $R = A \times A$ i.e., $R = A^2$. In other words, a relation R in a set A is universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

For example, *let* $A = \{1,2\}$. *Let* $R = \{(1,1),(1,2),(2,1),(2,2)\}$. *Here* $R = A \times A$, *so relation R is universal relation on A.*

<u>NOTE</u> The void relation *i.e.,* ϕ and universal relation *i.e.,* $A \times A$ on A are respectively the *smallest* and *largest* relations defined on the set A. Also these are sometimes called *Trivial Relations*. And, any other relation is called a *non-trivial relation*.

❖ **The relations $R = \phi$ and $R = A \times A$ are two *extreme relations*.**

c) Identity relation: A relation R on a set A is said to be the identity relation on A if $R = \{(a,b) : a \in A, b \in A \text{ and } a = b\}$.

Thus identity relation $R = \{(a,a) : \forall\ a \in A\}$.

The identity relation on set A is also denoted by I_A.

For example, *let* $A = \{1,2,3,4\}$. *Then* $I_A = \{(1,1),(2,2),(3,3),(4,4)\}$. *But the relation given by* $R = \{(1,1),(2,2),(1,3),(4,4)\}$ *is not an identity relation because element 1 is related to elements 1 and 3.*

<u>NOTE</u> In an identity relation on A every element of A should be related to itself only.

d) Reflexive relation: A relation R on a set A is said to be reflexive if $a R a\ \forall a \in A$ i.e., $(a,a) \in R\ \forall a \in A$.

For example, *let* $A = \{1,2,3\}$, *and* R_1, R_2, R_3 *be the relations given as* $R_1 = \{(1,1),(2,2),(3,3)\}$, $R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3)\}$ *and* $R_3 = \{(2,2),(2,3),(3,2),(1,1)\}$. *Here* R_1 *and* R_2 *are reflexive relations on A but* R_3 *is not reflexive as* $3 \in A$ *but* $(3,3) \notin R_3$.

<u>NOTE</u> The identity relation is always a reflexive relation but the opposite may or may not be true. As shown in the example above, R_1 is both identity as well as reflexive relation on A but R_2 is only reflexive relation on A.

e) Symmetric relation: A relation R defined on a set A is symmetric if $(a,b) \in R \Rightarrow (b,a) \in R\ \forall a,b \in A$ i.e., $a R b \Rightarrow b R a$ (i.e., whenever aRb then, bRa).

For example, *let* $A = \{1,2,3\}$, $R_1 = \{(1,2),(2,1)\}$, $R_2 = \{(1,2),(2,1),(1,3),(3,1)\}$, $R_3 = \{(2,3),(3,2),(2,2),(2,2)\}$ *i.e.* $R_3 = \{(2,3),(3,2),(2,2)\}$ *and* $R_4 = \{(2,3),(3,1),(1,3)\}$. *Here* R_1, R_2 *and* R_3 *are symmetric relations on A. But* R_4 *is not symmetric because* $(2,3) \in R_4$ *but* $(3,2) \notin R_4$.

f) Transitive relation: A relation R on a set A is transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ i.e., $a R b$ and $b R c \Rightarrow a R c$.

For example, *let* $A = \{1,2,3\}$, $R_1 = \{(1,2),(2,3),(1,3),(3,2)\}$ *and* $R_2 = \{(1,3),(3,2),(1,2)\}$. *Here* R_2 *is transitive relation whereas* R_1 *is not transitive because* $(2,3) \in R_1$ *and* $(3,2) \in R_1$ *but* $(2,2) \notin R_1$.

g) Equivalence relation: Let A be a non-empty set, then a relation R on A is said to be an equivalence relation if

(i) R is reflexive i.e. $(a,a) \in R\ \forall a \in A$ i.e., aRa.

(ii) R is symmetric i.e. $(a,b) \in R \Rightarrow (b,a) \in R\ \forall a,b \in A$ i.e., $aRb \Rightarrow bRa$.

(iii) R is transitive i.e. $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R\ \forall a,b,c \in A$ i.e., aRb and

$b\text{R}c \Rightarrow a\text{R}c$.

For example, *let* $A=\{1,2,3\}$, $R=\{(1,2),(1,1),(2,1),(2,2),(3,3)\}$. *Here R is reflexive, symmetric and transitive. So R is an equivalence relation on A.*

❖ ***Equivalence classes**:* Let A be an equivalence relation in a set A and let $a \in \text{A}$. Then, the set of all those elements of A which are related to a, is called equivalence class determined by a and it is denoted by $[a]$. Thus, $[a]=\{b \in \text{A} : (a, b) \in \text{A}\}$.

NOTE (i) Two equivalence classes are either disjoint or identical.
(ii) An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.

An important property of an equivalence relation is that it divides the set into pair-wise disjoint subsets called **equivalence classes** whose collection is called **a partition of the set**. Note that the union of all equivalence classes gives the whole set.
e.g. Let R denotes the equivalence relation in the set Z of integers given by R = {(a, b) : 2 divides $\text{a}-\text{b}$}. Then the equivalence class [0] is [0] = $\{0, \pm 2, \pm 4, \pm 6, \ldots\}$

06. TABULAR REPRESENTATION OF A RELATION

In this form of representation of a relation R from set A to set B, elements of A and B are written in the first column and first row respectively. If $(a,b) \in \text{R}$ then we write '1' in the row containing *a* and column containing *b* and if $(a,b) \notin \text{R}$ then we write '0' in the same manner.

For example, *let* $A=\{1,2,3\}, B=\{2,5\}$ *and* $R=\{(1,2),(2,5),(3,2)\}$ *then,*

R	***2***	***5***
1	*1*	*0*
2	*0*	*1*
3	*1*	*0*

07. INVERSE RELATION

Let $\text{R} \subseteq \text{A} \times \text{B}$ be a relation from A to B. Then, the inverse relation of R, to be denoted by R^{-1}, is a relation from B to A defined by $\text{R}^{-1}=\{(b,a):(a,b) \in \text{R}\}$.

Thus $(a,b) \in \text{R} \Leftrightarrow (b,a) \in \text{R}^{-1} \; \forall a \in \text{A}, b \in \text{B}$.

Clearly, $\text{Dom.}\left(\text{R}^{-1}\right)=\text{Range of R, Range of R}^{-1}=\text{Dom.}(\text{R})$.

Also, $\left(\text{R}^{-1}\right)^{-1}=\text{R}$.

For example, *let* $A=\{1,2,4\}, B=\{3,0\}$ *and let* $R=\{(1,3),(4,0),(2,3)\}$ *be a relation from A to B then,* $R^{-1}=\{(3,1),(0,4),(3,2)\}$.

Summing up all the discussion given above, here is a recap of all these for quick grasp:

01.	**a)** A relation R from A to B is an empty relation or void relation iff $\text{R}=\phi$.
	b) A relation R on a set A is an empty relation or void relation iff $\text{R}=\phi$.
02.	**a)** A relation R from A to B is a universal relation iff $\text{R}=\text{A} \times \text{B}$.
	b) A relation R on a set A is a universal relation iff $\text{R}=\text{A} \times \text{A}$.
03.	A relation R on a set A is reflexive iff $a\text{R}a, \forall a \in \text{A}$.
04.	A relation R on set A is symmetric iff whenever $a\text{R}b$, then $b\text{R}a$ for all $a,b \in \text{A}$.

05.	A relation R on a set A is transitive iff whenever $a\text{R}b$ and $b\text{R}c$, then $a\text{R}c$.
06.	A relation R on A is identity relation iff $\text{R}=\{(a,a),\forall a\in \text{A}\}$ i.e., R contains only elements of the type $(a,a)\forall a\in \text{A}$ and it contains no other element.
07.	A relation R on a non-empty set A is an equivalence relation iff the following conditions are satisfied: i) **R is reflexive** i.e., for every $a\in \text{A}$, $(a,a)\in \text{R}$ i.e., $a\text{R}a$. ii) **R is symmetric** i.e., for $a,b\in \text{A}$, $a\text{R}b\Rightarrow b\text{R}a$ i.e., $(a,b)\in \text{R}\Rightarrow (b,a)\in \text{R}$. iii) **R is transitive** i.e., for all $a,b,c\in \text{A}$ we have, $a\text{R}b$ and $b\text{R}c\Rightarrow a\text{R}c$ i.e., $(a,b)\in \text{R}$ and $(b,c)\in \text{R}\Rightarrow (a,c)\in \text{R}$.

EXERCISE FOR PRACTICE

Q01. Let $\text{N}=$ set of all natural numbers and $\text{R}=\{(x,y):x+2y=0;\,y\in \text{N}\}$. Is R a relation on N? Give the reason in support of your answer.

Q02. Let $\text{A}=\{2,4,5\}$, $\text{B}=\{1,2,3,4,6,8\}$ and let R be a relation from A to B defined by $x\text{R}y\Leftrightarrow x$ divides y. Find the relation (in roster form), its domain and range.

Q03. Let $\text{A}=\{1,2,3,4,6\}$ and let R be a relation on A defined by $\text{R}=\{(a,b):a,b\in \text{A};a \text{ divides } b\}$. Find the relation, its domain and range.

Q04. Determine the domain and range of the relation $\text{R}=\{(x,y):y=|x-1|,x\in \text{Z} \text{ and } |x|\le 3\}$.

Q05. Write the domain and range of relation $\text{R}=\{(x+1,x+5):x\in \{0,1,2,3,4,5\}\}$.

Q06. Let $\text{A}=\{1,2,3,4,5\}$ and $\text{B}=\{1,2,3,4,...,65\}$. Let R be a relation from A to B defined by $a\text{R}b$ iff *a is cube root of b.* Find R and its domain and range.

Q07. Let $\text{R}=\{(1,-1),(2,0),(3,1),(5,3)\}$. Find the inverse of R i.e., R^{-1} and its domain and range.

Q08. Let $\text{A}=\{1,2\}$. How many relations are possible on set A. List all of the relations.

Q09. Let $\text{A}=\{3,5\}$, $\text{B}=\{7,11\}$. Let $\text{R}_1=\{(a,b):a\in \text{A},b\in \text{B},\, a-b \text{ is odd}\}$ and $\text{R}_2=\{(a,b):a\in \text{A},b\in \text{B},\, a-b \text{ is even}\}$. Show that the relations R_1 and R_2 are respectively empty and universal relation from A into B.

Q10. Let $\text{A}=\{1,2,3\}$ and $\text{R}=\{(a,b):a,b\in \text{A},a \text{ divides } b \text{ and } b \text{ divides } a\}$. Show that R is an identity relation on A.

Q11. Let N be the set of all natural numbers and the relation R on N be defined by $x\text{R}y\Leftrightarrow x$ divides $y\ \forall\, x,y\in \text{N}$. Examine whether R is reflexive, symmetric or transitive.

Q12. Check whether the relation R defined in the set $\text{A}=\{1,2,3,4,5,6\}$ as $\text{R}=\{(a,b):b=a+1\}$ is reflexive, symmetric or transitive.

Q13. Show that the relation R in the set $\text{A}=\{1,2,3\}$ given by $\text{R}=\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$ is reflexive but neither symmetric nor transitive. Is it equivalence relation? Why?

Q14. Check if the relation R in the set $\text{A}=\{1,2,3,...,14\}$ defined by $\text{R}=\{(x,\,y):3x-y=0\}$ is reflexive, symmetric or transitive.

Q15. Determine whether the relation R which is defined in the set $\text{A}=\{1,2,3,4,5,6\}$ as $\text{R}=\{(x,y):y \text{ is divisible by } x\}$ is reflexive, symmetric or transitive.

Q16. Show that the relation R in the set Z of integers given by $\text{R}=\{(a,b):2 \text{ divides } a-b\}$ is an equivalence relation.

Q17. Show that the relation R on set $A=\{1,2,3,4,5\}$ given by $R=\{(a,b):|a-b| \text{ is even}\}$ is equivalence relation.

Q18. Let R be a relation on the set of all lines in a plane defined by $(l_1,l_2)\in R \Leftrightarrow l_1$ is parallel to l_2. Show that R is an equivalence relation.

Q19. Prove that the relation R on the set Z of all integers defined by $(x,y)\in R \Leftrightarrow x-y$ is divisible by n, is an equivalence relation on Z.

Q20. **a)** Let A = {1, 2, 3}. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is four.

b) Show that the number of equivalence relation in the set $\{1,2,3\}$ containing $(1,2)$ and $(2,1)$ is two.

Q21. Show that the relation R on the set $A=\{x\in Z: 0\le x\le 12\}$, given by $R=\{(a,b):|a-b| \text{ is a multiple of } 4\}$ is an equivalence relation.

Hence find the set of all elements related to 1 in R.

Q22. Let R be the relation in the set N given by $R=\{(a,b): a=b-2,\ b>6\}$. Which of the following is true?

a) $(2,4)\in R$ b) $(3,8)\in R$ c) $(6,8)\in R$ d) $(8,7)\in R$.

Q23. If R_1 and R_2 are equivalence relations in a set A, show that $R_1\cap R_2$ is also an equivalence relation.

Q24. Let T be the set of all triangles in a plane with R a relation in T given by R = {(T_1, T_2) : T_1 is congruent to T_2}. Show that R is an equivalence relation.

OR Let T be the set of all triangles in a plane with R as relation in T given by $R=\{(T_1,T_2): T_1\cong T_2\}$. Show that R is an equivalence relation.

Q25. Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is perpendicular to L_2}. Show that R is symmetric but neither reflexive nor transitive.

Q26. Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if $xv = yu$. Show that R is an equivalence relation.

Q27. Show that the relation R on the set R of real nos., defined as $R=\{(a,b): a\le b^2\}$ is neither reflexive nor symmetric nor transitive.

Q28. Let A = {1, 2, 3}. Find the number of equivalence relations containing (1, 2).

Q29. Show that the relation R in the set $A=\{x\in Z: 0\le x\le 12\}$ is an equivalence relation where R = { (a, b) : a = b}. Hence find the set of all elements related to 1 in R.

Q30. Let Z be the set of all integers and R be a relation on Z defined as $R=\{(a,b): a,b\in Z \text{ and } (a-b) \text{ is divisible by } 5\}$. Prove that R is an equivalence relation.

Q31. Show that the relation R in the set A = {1,2,3,4,5} given by R = {(a, b): $|a - b|$ is even} is an equivalence relation. Show that all the elements of {1,3,5} are related to each other and all the elements of {2,4} are related to each other. But no element of {1,3,5} is related to any element of {2,4}.

Q32. Let A = {1, 2, 3}. Find the number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive.

Q33. Let A = {1, 2, 3, …, 9} and R be the relation in $A\times A$ defined by (a, b) R (c, d) if a + d = b + c for (a, b), (c, d) in $A\times A$. Prove that R is an equivalence relation. Also obtain the equivalence class [(2, 5)].

Q34. (a) If R = {(x, y) : x + 2y = 8} is a relation on N, write the range of R.

(b) Let R be the equivalence relation in the set A = {0, 1, 2, 3, 4, 5} given by R = {(a, b) : 2 divides (a – b)}. Write the equivalence class [0].

(c) Write the smallest equivalence relation R on set A = {1, 2, 3}.

(d) Let R = {(a, a^3) : a is a prime number less than 5} be a relation. Find the range of R.

Q35. Determine whether the relation R defined on the set R of all real numbers as $R = \{(a, b): a, b \in R$ and $a - b + \sqrt{3} \in S$, where S is the set of all irrational numbers$\}$, is reflexive, symmetric and transitive.

Q36. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a,b)R(c,d)$ if $ad(b+c) = bc(a+d)$. Show that R is an equivalence relation.

FUNCTIONS & ITS VARIOUS TYPES

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. CONSTANT & TYPES OF VARIABLES

a) **Constant:** A constant is a symbol which retains the same value throughout a set of operations. So, a symbol which denotes a particular number is a constant. Constants are usually denoted by the symbols ***a, b, c, k, l, m,…*** etc.

b) **Variable:** It is a symbol which takes a number of values i.e., it can take any arbitrary values over the interval on which it has been defined. For example if x is a variable over R (set of real numbers) then we mean that x can denote any arbitrary real number. Variables are usually denoted by the symbols ***x, y, z, u, v,…*** etc.

c) **Independent variable:** That variable which can take an arbitrary value from a given set is termed as an independent variable.

d) **Dependent variable:** That variable whose value depends on the independent variable is called a dependent variable.

02. Defining A Function: Consider A and B be two non- empty sets then, a rule f which associates **each element of A with a unique element of B** is called a *function* or the *mapping from A to B* or *f maps A to B*. If f is a mapping from *A* to *B* then, we write $f: A \to B$ which is read as '*f is a mapping from A to B*' or '*f is a function from A to B*'.

If f associates $a \in A$ to $b \in B$, then we say that '***b* is the image of the element *a* under the function *f***' or '***b* is the *f* - image of *a***' or '**the value of *f* at *a***' and denote it by $f(a)$ and we write $b = f(a)$. The element a is called the **pre-image** or **inverse-image of *b***.

Thus for a function from A to B,

(i) A and B should be non-empty.
(ii) Each element of A should have image in B.
(iii) No element of A should have more than one image in B.
(iv) If A and B have respectively m and n number of elements then the **number of functions defined from A to B is n^m**.

03. Domain, Co-domain & Range of a function

The **set A is called the domain** of the function f and the **set B is called the co- domain**. The set of the images of all the elements of A under the function f is called the **range of the function *f*** and is denoted as $f(A)$.

Thus range of the function f is $f(A) = \{f(x): x \in A\}$.
Clearly $f(A) \subseteq B$.

☞ **Note the followings :**

(i) It is necessary that every f-image is in B; but there may be some elements in B which are not the f-images of any element of A i.e., whose pre-image under f is not in A.
(ii) Two or more elements of A may have same image in B.
(iii) $f: x \to y$ means that under the function f from A to B, an element x of A has image y in B.
(iv) Usually we denote the function f by writing $y = f(x)$ and read it as '***y* is a function of *x***'.

POINTS TO REMEMBER FOR FINDING THE DOMAIN & RANGE

Domain: If a function is expressed in the form $y = f(x)$, then domain of *f* means **set of all those real values of *x* for which *y* is real (i.e., *y* is well - defined)**.

❖ Remember the following points:

(i) *Negative number should not occur under the square root (even root) i.e., expression under the square root sign must be always* ≥ 0.

(ii) *Denominator should never be zero.*

(iii) *For* $\log_b a$ *to be defined* $a > 0$, $b > 0$ *and* $b \neq 1$. *Also note that* $\log_b 1$ *is equal to zero i.e.* 0.

Range: If a function is expressed in the form $y = f(x)$, then range of *f* means **set of all possible real values of *y* corresponding to every value of *x* in its domain**.

❖ Remember the following points:

(i) *Firstly find the domain of the given function.*

(ii) *If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.*

(iii) *If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.*

04. Function as a special type of relation: A relation *f* from a set A to another set B is said be a function (or mapping) from A to B if with every element (say *x*) of A, the relation *f* relates a unique element (say *y*) of B. This *y* is called *f*-image of *x*. Also *x* is called pre-image of *y* under *f*.

05. Difference between relation and function: A relation from a set A to another set B is any subset of $A \times B$; while a function *f* from A to B is a subset of $A \times B$ satisfying following conditions:

(i) For every $x \in A$, there exists $y \in B$ such that $(x, y) \in f$

(ii) If $(x, y) \in f$ and $(x, z) \in f$ then, $y = z$.

Sl. No.	Function	Relation
01.	Each element of A must be related to some element of B.	There may be some element of A which are not related to any element of B.
02.	An element of A should not be related to more than one element of B.	An element of A may be related to more than one elements of B.

06. Real valued function of a real variable: If the domain and range of a function *f* are subsets of R (the set of real numbers), then *f* is said to be **a real valued function of a real variable** or **a real function**.

07. Some important real functions and their domain & range

FUNCTION	REPRESENTATION	DOMAIN	RANGE
a) Identity function	$I(x) = x \ \forall x \in R$	R	R
b) Modulus function or Absolute value function	$f(x) = \lvert x \rvert = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$	R	$[0, \infty)$
c) Greatest integer function or Integral function or Step function	$f(x) = [x]$ or $f(x) = \lfloor x \rfloor \ \forall x \in R$	R	Z
d) Smallest integer function	$f(x) = \lceil x \rceil \ \forall x \in R$	R	Z
e) Signum function	$f(x) = \begin{cases} \frac{\lvert x \rvert}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ i.e., $f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$	R	$\{-1, 0, 1\}$

f) Exponential function	$f(x)=a^x \ \forall a \neq 1, a>0$	R	$(0,\infty)$
g) Logarithmic function	$f(x)=\log_a x \ \forall a \neq 1, a>0$ and $x>0$	$(0,\infty)$	R

08. TYPES OF FUNCTIONS

a) **One-one function (Injective function or Injection):** A function $f: \mathrm{A} \to \mathrm{B}$ is one- one function or injective function if distinct elements of A have distinct images in B.

Thus, $f: \mathrm{A} \to \mathrm{B}$ is one-one $\Leftrightarrow f(a)=f(b) \Rightarrow a=b \ \forall a,b \in \mathrm{A}$
$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \ \forall a,b \in \mathrm{A}$.

❖ If A and B are two sets having m and n elements respectively such that $m \le n$, then **total number of one-one functions** from set A to set B is ${}^n\mathrm{C}_m \times m!$ *i.e.*, ${}^n\mathrm{P}_m$.

❖ If n(A) = n then the number of injective functions defined from A onto itself is $n!$.

ALGORITHM TO CHECK THE INJECTIVITY OF A FUNCTION

STEP1- Take any two arbitrary elements *a, b* in the domain of *f*.

STEP2- Put $f(a)=f(b)$.

STEP3- Solve $f(a)=f(b)$. If it gives $a=b$ only, then *f* is a one-one function.

b) **Onto function (Surjective function or Surjection):** A function $f: \mathrm{A} \to \mathrm{B}$ is onto function or a surjective function if every element of B is the *f*-image of some element of A. That implies $f(\mathrm{A})=\mathrm{B}$ or range of *f* is the co-domain of *f*.

Thus, $f: \mathrm{A} \to \mathrm{B}$ is onto $\Leftrightarrow f(\mathrm{A})=\mathrm{B}$ *i.e.*, range of f = co-domain of f.

ALGORITHM TO CHECK THE SURJECTIVITY OF A FUNCTION

STEP1- Take an element $b \in \mathrm{B}$.

STEP2- Put $f(x)=b$.

STEP3- Solve the equation $f(x)=b$ for *x* and obtain *x* in terms of *b*. Let $x=g(b)$.

STEP4- If for all values of $b \in \mathrm{B}$, the values of *x* obtained from $x=g(b)$ are in A, then *f* is onto. If there are some $b \in \mathrm{B}$ for which values of *x*, given by $x=g(b)$, is not in A. Then *f* is not onto.

c) **One-one onto function (Bijective function or Bijection):** A function $f: \mathrm{A} \to \mathrm{B}$ is said to be one-one onto or bijective if it is both one-one and onto i.e., if the distinct elements of A have distinct images in B and each element of B is the image of some element of A.

❖ Also note that a **bijective function is also called a one-to-one function or one-to-one correspondence.**

❖ If $f: \mathrm{A} \to \mathrm{B}$ is a function such that,
i) f is one-one $\Rightarrow n(\mathrm{A}) \le n(\mathrm{B})$.
ii) f is onto $\Rightarrow n(B) \le n(\mathrm{A})$.
iii) f is one-one onto $\Rightarrow n(\mathrm{A}) = n(\mathrm{B})$.

❖ For an ordinary finite set A, a one-one function $f: \mathrm{A} \to \mathrm{A}$ is necessarily onto and an onto function $f: \mathrm{A} \to \mathrm{A}$ is necessarily one-one for every finite set A.

d) **Identity Function:** The function $I_A: \mathrm{A} \to \mathrm{A}$; $I_\mathrm{A}(x)=x \ \forall x \in \mathrm{A}$ is called an identity function on A.

NOTE Domain(I_A) = A and Range(I_A) = A.

e) **Equal Functions:** Two function *f* and *g* having the same domain *D* are said to be equal if $f(x)=g(x)$ for all $x \in D$.

09. INVERSE OF A FUNCTION

Let $f : A \to B$ be a bijection. Then a function $g : B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of *f* i.e., $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of *f* is generally denoted by f^{-1}.

Thus, if $f : A \to B$ is a bijection, then a function $f^{-1} : B \to A$ is such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

ALGORITHM TO FIND THE INVERSE OF A FUNCTION

STEP1- Put $f(x) = y$ where $y \in B$ and $x \in A$.

STEP2- Solve $f(x) = y$ to obtain x in terms of y.

STEP3- Replace x by $f^{-1}(y)$ in the relation obtained in STEP2.

STEP4- In order to get the required inverse of f i.e. $f^{-1}(x)$, replace y by x in the expression obtained in STEP3 i.e. in the expression $f^{-1}(y)$.

EXERCISE FOR PRACTICE

Q01. Check whether $f : R \to R$ given as $f(x) = x^3 + 2$ for all $x \in R$ is one- one or not.

Q02. Discuss the surjectivity of $f : Z \to Z$ given as $f(x) = 3x + 2 \quad \forall x \in Z$.

Q03. Prove that $f : Q \to Q$ given by $f(x) = 2x - 3 \;\; \forall x \in Q$ is a bijection.

Q04. Show that $f : N \to N$ given by $f(x) = 2x$ is one- one but not onto.

Q05. Show that $f : R \to R$ given by $f(x) = x^2$ is neither one- one nor onto.

Q06. For the function $f : R \to R$ given by $f(x) = 2x$, prove that the function f is one- one and onto both. Is it a bijection?

Q07. Show that $f : R \to R$ given by $f(x) = ax + b$ where $a, b \in R,\ a \neq 0$ is a bijection.

Q08. Show that $f : R \to R$ given by $f(x) = 5x^3 + 4$ is bijective.

Q09. Let $A = R - \{2\}$, $B = R - \{1\}$. If $f : A \to B$ is a mapping defined by $f(x) = \frac{x-1}{x-2}$ then, show that f is bijection.

Q10. Show that the function $f : N \to N$ given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for all $x > 2$ is onto but not one-one.

Q11. **a)** Show that $f : N \to N$ given by $f(x) = \begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$ is both one- one and onto.

b) Show that $f : N \cup \{0\} \to N \cup \{0\}$ given by $f(n) = \begin{cases} n+1, \text{ if } n \text{ is even} \\ n-1, \text{ if } n \text{ is odd} \end{cases}$ is bijection. Also show that $f^{-1} = f$.

c) Let $f : W \to W$, be defined as $f(x) = x - 1$, if x is odd and $f(x) = x + 1$, if x is even. Show that *f* is invertible. Find the inverse of *f*, where W is the set of all whole numbers.

Q12. **a)** Show that an onto function $f : \{1,2,3\} \to \{1,2,3\}$ is always one-one.

b) Show that a one-one function $f : \{1,2,3\} \to \{1,2,3\}$ must be onto.

Q13. Show that the signum function $f : R \to R$ given by $f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$ is neither one-one nor onto.

Q14. Let $A = R - \{3\}$ and $B = R - \{1\}$, consider the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$. Is f one-one and onto? Give reasons.

Q15. Let $A = R - \{3\}$ and $B = R - \{1\}$, consider the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$. Is f one-one and onto? Give reasons.

Q16. Show that if $f : R - \left\{\frac{7}{5}\right\} \to R - \left\{\frac{3}{5}\right\}$ is defined by $f(x) = \frac{3x+4}{5x-7}$ & $g : R - \left\{\frac{3}{5}\right\} \to R - \left\{\frac{7}{5}\right\}$ is defined by $g(x) = \frac{7x+4}{5x-3}$ then, $fog = I_A$ & $gof = I_B$, where $A = R - \left\{\frac{3}{5}\right\}$, $B = R - \left\{\frac{7}{5}\right\}$; $I_A(x) = x \ \forall x \in A$, $I_B(x) = x \ \forall x \in B$ are called the **identity functions** on the sets A and B respectively.

Q17. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$ where, $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$. Show that f is invertible. Find the inverse of f.

Q18. Let $Y = \{n^2 : n \in N\} \subset N$. Consider $f : N \to Y$ as $f(n) = n^2$. Show that f is invertible and if the inverse exists, find it.

Q19. Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to S$ where, S is the range of f, is invertible. Find the inverse of f.

Q20. If $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$, show that $fof(x) = x$ for all $x \neq \frac{2}{3}$. Write the expression for f^{-1}.

Q21. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible such that $f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$.

Q22. Let $f : R \to R$ be given by $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$. Is f a one- one function?

Q23. If $f(x) = \frac{1}{2x+1}$, $x \neq -\frac{1}{2}$ then, show that $f(f(x)) = \frac{2x+1}{2x+3}$, $x \neq -\frac{1}{2}, -\frac{3}{2}$.

Q24. If $f : R \to R$ is a bijection given by $f(x) = x^3 + 3$ then, find $f^{-1}(x)$.

Q25. Check whether $f : R - \{-1\} \to R - \{1\}$ defined by $f(x) = \frac{x}{x+1}$ is invertible. If it is invertible then, find f^{-1}.

Q26. If $A = \{0, 1, 2, 3\}, B = \{7, 9, 11, 13\}$ and a rule f from A to B is defined by $f(x) = 2x + 7 \ \forall x \in A$, then prove that f is one-one and onto.

Q27. Let $A = \{1, 2, 3\}, B = \{2, 4, 6\}$. If $f : A \to B$ is a function defined as $f(1) = 2$, $f(2) = 4$, $f(3) = 6$. Write down f^{-1} as a set of ordered pairs.

Q28. Let $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is a one-one function.

Q29. If the function $f : R \to (0, 2)$ defined by $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1$ is invertible then, find $f^{-1}(x)$.

Q30. Consider a function $f : [0, \pi/2] \to R$ given by $f(x) = \sin x$ and $g : [0, \pi/2] \to R$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one- one.

Q31. Let $f : R \to R$ be defined as $f(x) = \cos(5x + 2)$. Is f invertible? Justify your answer.

Q32. Find the number of all one-one functions from set A = {1, 2, 3} to itself.

Q33. **a)** If X and Y are two sets having 2 and 3 elements respectively then, find the number of functions from X to Y.

b) If A = {1, 2, 3} and B = {a, b}, write the total number of function from A to B.

c) If A = {a, b, c} and B = {–2, –1, 0, 1, 2}, write the total number of one-one functions defined from A to B.

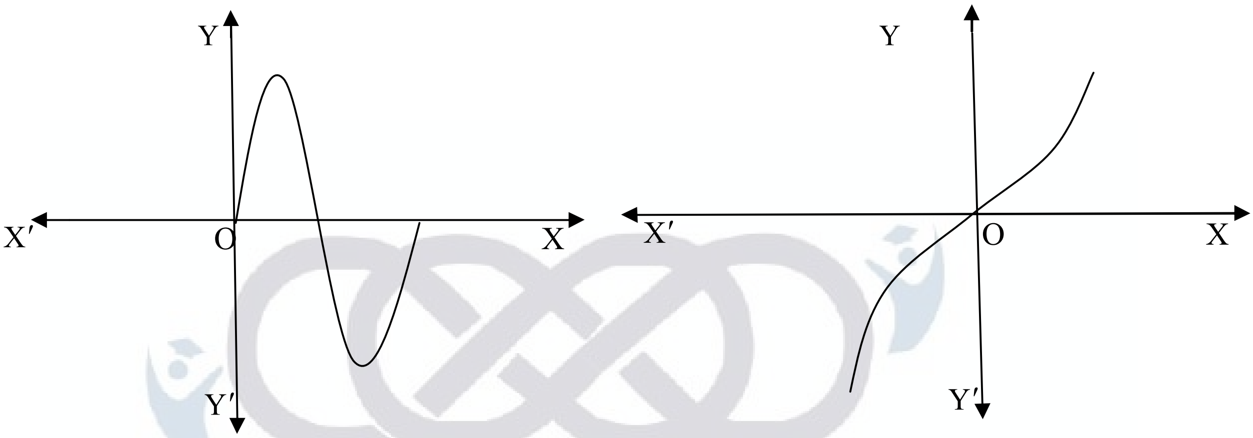
d) Write the total number of one-one functions from {1, 2, 3, 4} to {a, b, c}.

e) Find the number of all onto functions from the set {1, 2, 3, ..., n} to itself.

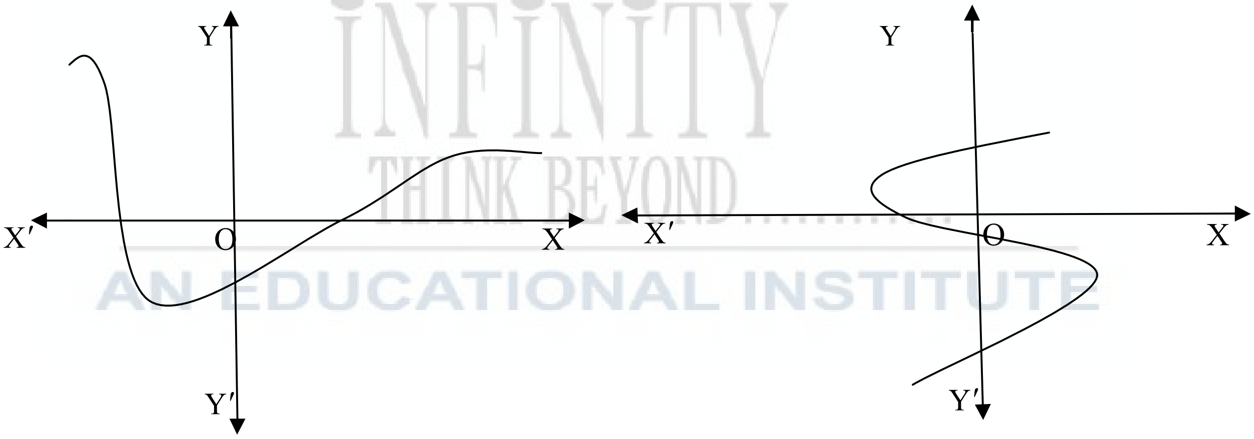
Q34. What is the range of the function $f(x) = \frac{|x-1|}{(x-1)}$?

Q35. Which one graph of the Fig.2 represents a one-one function?

Q36. Which one graph of the Fig.1 represents a function?



(a) Fig.1 (b)



(a) Fig.2 (b)

Q37. Let $f : \mathrm{N} \to \mathrm{N}$ be defined as $f(n) = \begin{cases} \frac{n+1}{2}, & \text{when } n \text{ is odd} \\ \frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$ for all $n \in \mathrm{N}$.

State whether the function f is bijective. Justify your answer.

Q38. Consider $f : R_+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \frac{\sqrt{54+5y}-3}{5}$.

Q39. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that *f* is invertible with the inverse f^{-1} of *f* given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

ADDITIONAL QUESTIONS

Q1. Show that the relation R on the set A = Show that the relation R on the set A = $\{x \in Z : 0 \leq x \leq 12\}$, given by R = {(a, b) : |a – b| is multiple of 4} is an equivalence relation.

Q2. Let Z be the set of all integers and R be the relation on Z defined as R = {(a,b) : a ,b ∈ z and (a -b) is divisible by 5}. Prove that R is an equivalence relation.

Q3. Show that the relation S in the set R of real numbers, defined as s ={ (a,b) : a,b ∈ R and $a \leq b^2$ }

Q4. Show that the relation R in the set A = {1,2,3,4,5} given by R = {(a,b) : |a – b| is even }, is and all the equivalence relation. Show that all the elements of {1,3,5} are related to each other and all the elements of {2,4} are related to each other.But no elements of {1,3,5} is related to any element of {2,4}.

Q5. Let R be relation defined on the set of natural numbers N as follows:
R = {(x , y) : x ∈ N , y ∈ N and 2x+y = 24 }. Find the domains and range of the relation R. Also find if R is an equivalence relation or not.

Q6. Let A = {1,2,3,…9} and R be relation in A × A defined by (a,b) R (c,d) if a + d = b + c for (a,b) , (c,d) in A × A. Prove that R is an equivalence relation. Also obtain the equivalence class {(2,5)}.

Q7. Let N denote the set of all natural numbers and R be the relation on N × N defined by (a,b) R (c,d) if ad(b + c) = bc (a + d). Show that R is an equivalence relation.

Q8. Let f: N → R be a function defined as $f(x) = 4x^2 + 12x + 15$ .Show that f : N → S , where S is the range of f , is invertible. Also find the inverse of f.

Q9. Determine whether the relation R defined on the set R of all real numbers as R = {(a,b) : a, b +$\sqrt{3}$ ∈ S ,where S is the set of all irrational numbers } , is reflexive , symmetric and transitive.

Q10. Consider f: R → (9,∞) given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \frac{\sqrt{54+5y}-3}{5}$.

Q11. Let f : W →W be defined as $f(n) = \begin{cases} n-1, if\ n\ is\ odd \\ n+1, if\ n\ is\ even \end{cases}$

Show that f is invertible and find the inverse of f . Here, W is the set of all whole numbers.

Q12. Show that the relation R defined by (a,b) R (c,d) ⇒a+d =b +c on the A × A , where A = {1,2.3,…,10} is an equivalence relation.

Q13. Show that the function f: R → {x ∈ R : -1 < x < 1} defined by $f(x) = \frac{x}{1+x}$, x ∈ R is one-one and onto function . Here find $f^{-1}(x)$.

Q14. Let f: R - $\left\{-\frac{4}{3}\right\}$ → R be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that , in f: R - $\left\{-\frac{4}{3}\right\}$ → Range of f , f is one- one and onto. Hence find f^{-1} in range f → R- $\left\{-\frac{4}{3}\right\}$.

Q15. Let A = {x ∈ z : 0 ≤ x ≤ 12 }. Show that R = {(a,b) : a,b ∈ A ,|a – b| is divisible by 4 } is an equivalence relation.
Find the set of all elements related to 1. Also write the equivalence class [2].

Important Notes

Important Notes

Important Notes

Important Notes